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A Fractional Calculus Generalization of Integer

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Abstract:
Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary. Differentiation and integration are usually regarded as discrete operations, in the sense that we differentiate or integrate a function once, twice, or any whole number of times. However, in some circumstances it's useful to evaluate a fractional derivative.

Key words: Fractional, Calculus, Integer, Derivation, Function

Introduction

The calculus of variations is a mathematical research field that was born in 1696 with the solution to the brachistochrone problem (see, e.g., [1]) and is focused on finding external values of functional [2, 3, 4, 1]. Usually, considered functional are given in the form of an integral that involves an unknown function and its derivatives. The calculus of variations possesses also important connections with other fields of mathematics, e.g., with the particularly important in this work fractional calculus.

Fractional calculus, i.e., the calculus of non-integer order derivatives, has also its origin in the 1600s. It is a generalization of (integer) differential calculus, allowing to define derivatives (and integrals) of real or complex order [5, 6, 7]. , in the last few decades, fractional problems have received an increasing attention of many researchers. As mentioned in [8], Science Watch of Thomson Reuters identified the subject as an Emerging Research Front area. Fractional derivatives are non-local operators and are historically applied in the study of non-local or time dependent processes [6]. The first and well established application of fractional calculus Physics was in the frame work of anomalous diffusion, hitches related to features observed in many physical systems.

Here we can mention the report [9] demonstrating that fractional equation work as a complementary tool in the description of anomalous transport processes. Within the fractional approach it is possible to include external fields in a straight forward manner As a consequence, in a short period of time the list of applications expanded. Applications include chaotic dynamics [10], material sciences [11], mechanics of fractal and complex media [12, 13], quantum mechanics [14, 15], physical kinetics [16],long-range dissipation[17],long-range interaction [18,19], just to mention a few. This diversity of applications makes the fractional calculus an important subject, which requires serious attention and strong interest.

The calculus of variations and the fractional calculus are connected since the XIX century. Indeed, in 1823 Niels Heinrik Abel applied the fractional calculus to the solution of an integral equation that arises in the formulation of the tautochrone problem. The fractional calculus of variations consists in extremizing (minimizing or maximizing) functional whose Lagrangians contain fractional integrals and derivatives. It was born in 1996-97, when Riewe derived Euler-Lagrange fractional differentiable equations and showed how non-conservative systems in mechanics can be described using fractional derivatives. A more general unifying perspective to the subject is, however, possible, by fractional operators depending on general kernels [20].In this work we follow such an approach, developing a generalized fractional calculus of variations. We consider problems, where the Lagrangians depend not only on classical derivatives but also on generalized fractional operators. Moreover, we discuss even more general problems, where also classical integrals are substituted by generalized fractional integrals and obtain general theorems, for several types of variational problems, which are valid for rather arbitrary operators and kernels. As special cases, one obtains there cent results available in the literature of fractional variational calculus [21].

1. Fractional Calculus

Fractional calculus is a generalization of (integer) differential calculus, in the sense that it deals with derivatives of real or complex order. It was introduced on 30th September 1695. On that day, Leibniz wrote a letter to L'Hopital, raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives. L'Hopital wanted to know the result for the derivative of order n = 1/2. Leibniz replied that one day, useful consequences will be drawn" and, in fact, his vision became areality.1819, when Lacroix presented a definition of fractional derivative based on the

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usual expression for the nth derivative of the power function [22]. Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., on-integer) differential operators were introduced: [23] and the more recent notions of Cresson [24], variable order fractional operators introduced by Samko and Ross in 1993 [25].

In 2010, an interesting perspective to the subject, unifying all mentioned notion so fractional derivatives and integrals, was introduced in [26] Precisely, authors considered general operators, which by choosing special kernels reduce to the standard fractional operators. However, other non standard kernels can also be considered as particular cases.

1.1 One-dimensional Fractional

We begin with basic facts on the one-dimensional classical, variable order, and generalized fractional operators.

1.1.1 Classical Fractional Operators

In this section, we present definitions and properties of the one-dimensional fractional integrals and derivatives under consideration. The reader interested in the subject is referred to the books [27].

Definition 1 (Left and right Hadamark fractional integrals). We define the left-sided and right-sided Hadamard integrals of fractional order $\alpha \in \mathbb{R}$ (α>0) by

$$ {}_aI_t^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \quad t \in (a, b] \qquad \text{............ (1)} $$

And

$$ {}_tI_b^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)d\tau}{(\tau-t)^{1-\alpha}}, \quad t \in [a, b) \qquad \text{................ (2)} $$

respectively.

Definition 2 (Left and right Riemann-Liouville fractional derivatives). The left Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}$ ($0 < \alpha < 1$) of a function f, denoted by ${}_\alpha D_t^\alpha[f]$, is defined by

$$ \forall t \in (a, b],\ {}_\alpha D_t^\alpha[f](t) := \frac{d}{dt}{}_\alpha I_t^{1-\alpha}[f](t) \text{.................(3)} $$

Similarly, the right Riemann-Liouville fractional derivative of order α of a function f, denoted by ${}_tD_b^\alpha[f]$, is defined by:

$$ \forall t \in (a, b],\ {}_tD_b^\alpha[f](t) := -\frac{d}{dt}{}_tI_b^{1-\alpha}[f](t) \text{................(4)} $$

As we can see below, Riemann-Liouville fractional integral and differential operators of power functions return power functions.

Property 1: (cf. Property 2.1 [51]). Now, let $1 > \alpha$, $\beta > 0$. Then the following identities hold:

$$ {}_aI_t^\alpha\left[(\Gamma - a)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} - (t-a)^{\beta+\alpha-1}, $$

$$ {}_aD_t^a\left[(\Gamma - a)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} - (t-a)^{\beta-\alpha-1}, $$

$$ {}_tI_b^\alpha\left[(b-\Gamma)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1}, $$

And

$$ {}_tD_b^\alpha\left[(b-\Gamma)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1} \text{............................(5)} $$

Property 2: (cf. Lemma 2.3 [51]). Let $>\alpha$, $\beta > 0$ and $f \in L^\Gamma(a, b; \mathbb{R})$, ($1 \le r \le \infty$). Then, equations

$$ \left({}_aI_t^\alpha \circ {}_aI_t^\beta\right)[f](t) = {}_aI_t^{\alpha+\beta}[f](t) $$

And

$$\left({}_tI_b^{\alpha} \circ {}_tI_b^{\beta}\right)[f](t) = {}_tI_b^{\alpha+\beta}[f](t) \quad .... (6)$$

Are satisfied

Next results show that, for certain classes of functions, Riemann-Liouville fractional derivatives and Caputo fractional derivatives are left inverse operators of Riemann-Liouville fractional integrals.

Property 3: (cf. Lemma 2.21 [51]). Let $1 > \alpha > 0$. If f is continuous on the interval [a, b], then

$$\left({}_aD_t^{\alpha} \circ {}_aI_t^{\alpha}\right)[f](t) = f(t)$$

$$\left({}_tD_b^{\alpha} \circ {}_tI_b^{\alpha}\right)[f](t) = f(t) \quad .... (7)$$

For r-Lebesgue integrable functions, Riemann-Liouville fractional integrals and derivatives satisfy the following composition properties.

Property 4: (cf. Lemma 2.4[51]). If $1 > \alpha > 0$ and $f \in L^r(a, b; \mathbb{R})$, $(1 \leq r \leq \infty)$, then the following is true:

$$\left({}_a^CD_t^{\alpha} \circ {}_aI_t^{\alpha}\right)[f](t) = f(t)$$

$$\left({}_t^CD_b^{\alpha} \circ {}_tI_b^{\alpha}\right)[f](t) = f(t) \quad .... (8)$$

Property 5: (cf. Property 2.2 [51]). Let $1 > \alpha > \beta > 0$ and $f \in L^r(a, b; \mathbb{R})$, $(1 \leq r \leq \infty)$, Then, relations

$$\left({}_aD_t^{\beta} \circ {}_aI_t^{\alpha}\right)[f](t) = {}_aI_t^{\alpha-\beta}[f](t)$$

And

$$\left({}_tD_b^{\beta} \circ {}_tI_b^{\alpha}\right)[f](t) = {}_tI_b^{\alpha-\beta}[f](t) \quad .... (9)$$

Are satisfied.

In classical calculus, integration by parts formula relates the integral of a product of functions to the integral of their derivative and anti-derivative. As we can see below, this formula works also for fractional derivatives, however it changes the type of differentiation: left Riemann-Lioville fractional derivatives are transformed to right Caputo fractional derivatives.

2. Variable Order Fractional Operators

In 1993, Samko and Ross [25] proposed an interesting generalization of fractional operators. They introduced the study of fractional integration and differentiation when the order is not a constant but a function. Several works were dedicated to variable order fractional operators, their applications and interpretations [28]. The paper [29] is devoted to the study of a variable-order fractional differential equation that characterizes some problems in the theory of viscoelasticity. The work [30] investigates the drag force acting on a particle due to the oscillatory flow a viscous fluid. In [31] a variable order differential equation for a particle in a quiescent viscous liquid is developed. For more on the application of variable order fractional operators to the modeling of dynamic systems, we refer to reader to the recent review article [32].

Let us introduce the following triangle:

$\Delta := \{(t,\Gamma) \in \mathbb{R}^2 : a \leq \Gamma < t \leq b\}$,

And let $\alpha(t,\Gamma): \Delta \to [0, 1]$ be such that $\alpha \in C^1(\Delta; \mathbb{R})$.

Definition 3 (Left and right Caputo derivatives of variable fractional order). The left Caputo derivative of variable fractional order $\alpha(.,.)$ is defined by

$$\forall t \in (a, b],\ {}_a^CD_t^{\alpha(.,.)}[f](t) := {}_aI_t^{1-\alpha(.,.)}\left[\frac{d}{dt}f\right](t).$$

While the right Caputo derivative of variable fractional order $\alpha(.,.)$ is given by

$$\forall t \in (a, b],\ {}_t^CD_b^{\alpha(.,.)}[f](t) := {}_tI_b^{1-\alpha(.,.)}\left[\frac{d}{dt}f\right](t) \quad ....(10)$$

Definition 4 (Left and right Riemann-Liouville derivatives of variable order). The left Riemann-Liouville derivative of variable fractional order α (.,.) of a function f is defined by

$$\forall t \in (a,b], \quad {}_aD_t^{\alpha(\cdot,\cdot)}[f](t) := \frac{d}{dt}\,{}_aI_t^{1-\alpha(\cdot,\cdot)}[f](t)$$

While the right Riemann-Liouville derivative of variable fractional order α (.,.) is defined by

$$\forall t \in [a,b), \quad {}_tD_b^{\alpha(\cdot,\cdot)}[f](t) := -\frac{d}{dt}\,{}_tI_b^{1-\alpha(\cdot,\cdot)}[f](t) \qquad \text{.......... (11)}$$

3. Generalized Fractional Operators

This section presents definitions of one-dimensional generalized fractional operators. In special cases, these operators simplify to the classical Riemann-Liouville fractional integrals, and Riemann-Liouville and Caputo fractional derivatives. As before,

$$\Delta = \{(t,\tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}.$$

Definition 5 (Generalized fractional derivative of Riemann-Liouville type). The generalized fractional derivative of Riemann-Liouville type, denoted by A_P, is defined by

$$A_P = \frac{d}{dt} \circ K_P.$$

The next differential operator is obtained by interchanging the order of the operators in the composition that defines A_P.

Definition 6 (Generalized fractional derivative of Caputo type). The general kernel differential operator of Caputo type, denoted by B_P, is given by

$$B_P = K_P \circ \frac{d}{dt}.$$

Example 3. The standard Riemann-Liouville and Caputo fractional derivatives (see, e.g., [51, 53, 91, 98] are easily obtained from the general kernel operators A_P and B_P, respectively. Let

$k^{\alpha}(t-\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha}$, $\alpha \in (0, 1)$. If P = <a, t, b, 1, 0>, then

$$A_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau)\,d\tau =: {}_aD_t^{\alpha}[f](t)$$

is the standard left Riemann-Liouville fractional derivative of orderα, while

$$B_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau)\,d\tau =: {}_a^CD_t^{\alpha}[f](t)$$

is the standard left Caputo fractional derivative of orderα; if P = <a, t, b, 0, 1>, then

$$-A_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} f(\tau)\,d\tau =: {}_tD_b^{\alpha}[f](t)$$

is the standard right Riemann-Liouville fractional of order α, while

$$-B_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} f'(\tau)\,d\tau = {}_t^CD_b^{\alpha}[f](t)$$

This is the standard right Caputo fractional derivative of orderα.

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